SCIENCE

FRIDAY, SEPTEMBER 12, 1913

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THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

THE PLACE OF PURE MATHEMATICS[1]

IT is not a very usual thing for the opening address of this section to be entrusted to one whose main energies have been devoted to what is called pure mathematics; but I value the opportunity in order to try to explain what, as I conceive it, the justification of the pure mathematician is. You will understand that in saying this I am putting myself in a position which belongs to me as little by vocation as by achievement, since it was my duty through many years to give instruction in all the subjects usually regarded as mathematical physics, and it is still my duty to be concerned with students in these subjects. But my experience is that the pure mathematician is apt to be regarded by his friends as a trifler and a visionary, and the consciousness of this becomes in time a paralyzing deadweight. I think that view is founded on want of knowledge.

Of course, it must be admitted that the mathematician, as such, has no part in those public endeavors that arise from the position of our empire in the world, nor in the efforts that must constantly be made for social adjustment at home. I wish to make this obvious remark. For surely the scientific man must give his time and his work in the faith of at least an intellectual harmony in things; and he must wish to know what to think of all that seems out of gear in the working of human relations. His

[1] Address of the president to the Mathematical and Physical Science Section of the British Association for the Advancement of Science, Birmingham, 1913.

own cup of contemplation is often golden; he marks that around him there is fierce fighting for cups that are earthen, and largely broken; and many there are that go thirsting. And, again, the mathematician is as sensitive as others to the marvel of each recurring springtime, when, year by year, our common mother seems to call us so loudly to consider how wonderful she is, and how dependent we are, and he is as curious as to the mysteries of the development of living things. He can draw inspiration for his own work, as he views the spectacle of a starry night, and sees

> How the floor of heaven
> Is thick inlaid with patines of bright gold.
> Each orb, the smallest, in his motion, sings,

but the song, once so full of dread, how much it owes to the highest refinements of his craft, from at least the time of the Greek devotion to the theory of conic sections; how much, that is, to the harmony that is in the human soul. Yet the mathematician bears to the natural observer something of the relation which the laboratory botanist has come to bear to the field naturalist. Moreover, he is shut off from inquiries which stir the public imagination; when he looks back the ages over the history of his own subject the confidence of his friends who study heredity and teach eugenics arouses odd feelings in his mind; if he feels the fascination which comes of the importance of such inquiries, he is also prepared to hear that the subtlety of nature grows with our knowledge of her. Doubtless, too, he wishes he had some participation in the discovery of the laws of wireless telegraphy, or had something to say in regard to the improvement of internal-combustion engines or the stability of aeroplanes; it is little compensation to remember, though the mathematical physicist is his most tormenting critic, what those of his friends who have the physical instinct used to say on the probable development of these things, however well he may recall it.

But it is not logical to believe that they who are called visionary because of their devotion to creatures of the imagination can be unmoved by these things. Nor is it at all just to assume that they are less conscious than others of the practical importance of them, or less anxious that they should be vigorously prosecuted.

Why is it, then, that their systematic study is given to other things, and not of necessity, and in the first instance, to the theory of any of these concrete phenomena? This is the question I try to answer. I can only give my own impression, and doubtless the validity of an answer varies as the accumulation of data, made by experimenters and observers, which remains unutilized at any time.

The reason, then, is very much the same as that which may lead a man to abstain from piecemeal, indiscriminate charity in order to devote his attention and money to some well-thought-out scheme of reform which seems to have promise of real amelioration. One turns away from details and examples, because one thinks that there is promise of fundamental improvement of methods and principles. This is the *argumentum ad hominem*. But there is more than that. The improvement of general principles is arduous, and if undertaken only with a view to results may be ill-timed and disappointing. But as soon as we consciously give ourselves to the study of universal methods for their own sake another phenomenon appears. The mind responds, mastery of the relations of things, hitherto unsuspected, begin to appear on the mental horizon. I am well enough aware of the retort to which such a statement is open. But, I say, interpret the fact as you will, our intellectual pleasure in life cometh not by might nor by power—arises, that is,

most commonly, not of set purpose—but lies at the mercy of the response which the mind may make to the opportunities of its experience. When the response proves to be of permanent interest—and for how many centuries have mathematical questions been a fascination?—we do well to regard it. Let us compare another case which is, I think, essentially the same. It may be that early forms of what now is specifically called art arose with a view to applications: I do not know. But no one will deny that art, when once it has been conceived by us, is a worthy object of pursuit; we know by a long trial that we do wisely to yield ourselves to a love of beautiful things, and to the joy of making them. Well, pure mathematics, as such, *is* an art, a creative art. If its past triumphs of achievement fill us with wonder, its future scope for invention is exhaustless and open to all. It is also a science. For the mind of man is one: to scale the peaks it spreads before the explorer is to open ever new prospects of possibility for the formulation of laws of nature. Its resources have been tested by the experience of generations; to-day it lives and thrives and expands and wins the life-service of more workers than ever before.

This, at least, is what I wanted to say, and I have said it with the greatest brevity I could command. But may I dare attempt to carry you further? If this seems fanciful, what will you say to the setting in which I would wish to place this point of view? And yet I feel bound to try to indicate something more, which may be of wider appeal. I said a word at starting as to the relations of science to those many to whom the message of our advanced civilization is the necessity, above all things, of getting bread. Leaving this aside, I would make another reference. In our time old outlooks have very greatly changed; old hopes, disregarded perhaps because undoubted, have very largely lost their sanction, and given place to earnest questionings. Can any one who watches doubt that the courage to live is in some danger of being swallowed up in the anxiety to acquire? May it not be, then, that it is good for us to realize, and to confess, that the pursuit of things that are beautiful, and the achievement of intellectual things that bring the joy of overcoming, is at least as demonstrably justifiable as the many other things that fill the lives of men? May it not be that a wider recognition of this would be of some general advantage at present? Is it not even possible that to bear witness to this is one of the uses of the scientific spirit? Moreover, though the pursuit of truth be a noble aim, is it so new a profession; are we so sure that the ardor to set down all the facts without extenuation is, unassisted, so continuing a purpose? May science itself not be wise to confess to what is its own sustaining force?

Such, ladies and gentlemen, in crude, imperfect phrase, is the *apologia*. If it does not differ much from that which workers in other ways would make, it does, at least, try to represent truly one point of view, and it seems to me specially applicable to the case of pure mathematics. But you may ask: What, then, is this subject? What can it be about if it is not primarily directed to the discussion of the laws of natural phenomena? What kind of things are they that can occupy alone the thoughts of a lifetime? I propose now to attempt to answer this, most inadequately, by a bare recital of some of the broader issues of present interest—though this has difficulties, because the nineteenth century was of unexampled fertility in results and suggestions, and I must be as little technical as possible.

PRECISION OF DEFINITIONS

First, in regard to two matters which illustrate how we are forced by physical problems into abstract inquiries. It is a constantly recurring need of science to reconsider the exact implication of the terms employed; and as numbers and functions are inevitable in all measurement, the precise meaning of number, of continuity, of infinity, of limit, and so on, are fundamental questions; those who will receive the evidence can easily convince themselves that these notions have many pitfalls. Such an imperishable monument as Euclid's theory of ratio is a familiar sign that this has always been felt. The last century has witnessed a vigorous inquiry into these matters, and many of the results brought forward appear to be new; nor is the interest of the matter by any means exhausted. I may cite, as intelligible to all, such a fact as the construction of a function which is continuous at all points of a range, yet possesses no definite differential coefficient at any point. Are we sure that human nature is the only continuous variable in the concrete world, assuming it be continuous, which can possess such a vacillating character? Or I may refer to the more elementary fact that all the rational fractions, infinite in number, which lie in any given range, can be enclosed in intervals whose aggregate length is arbitrarily small. Thus we could take out of our life all the moments at which we can say that our age is a certain number of years, and days, and fractions of a day, and still have appreciably as long to live; this would be true, however often, to whatever exactness, we named our age, provided we were quick enough in naming it. Though the recurrence of these inquiries is part of a wider consideration of functions of complex variables, it has been associated also with the theory of those series which Fourier used so boldly, and so wickedly, for the conduction of heat. Like all discoverers, he took much for granted. Precisely how much is the problem. This problem has led to the precision of what is meant by a function of real variables, to the question of the uniform convergence of an infinite series, as you may see in early papers of Stokes, to new formulation of the conditions of integration and of the properties of multiple integrals, and so on. And it remains still incompletely solved.

CALCULUS OF VARIATIONS

Another case in which the suggestions of physics have caused grave disquiet to the mathematicians is the problem of the variation of a definite integral. No one is likely to underrate the grandeur of the aim of those who would deduce the whole physical history of the world from the single principle of least action. Every one must be interested in the theorem that a potential function, with a given value at the boundary of a volume, is such as to render a certain integral, representing, say, the energy, a minimum. But in that proportion one desires to be sure that the logical processes employed are free from objection. And, alas! to deal only with one of the earliest problems of the subject, though the finally sufficient conditions for a minimum of a simple integral seemed settled long ago, and could be applied, for example, to Newton's celebrated problem of the solid of least resistance, it has since been shown to be a general fact that such a problem can not have any definite solution at all. And, although the principle of Thomson and Dirichlet, which relates to the potential problem referred to, was expounded by Gauss, and accepted by Riemann, and remains to-day in our standard treatise on National Philosophy, there can be no doubt that, in the form in which it was originally

stated, it proves just nothing. Thus a new investigation has been necessary into the foundations of the principle. There is another problem, closely connected with this subject, to which I would allude: the stability of the solar system. For those who can make pronouncements in regard to this I have a feeling of envy; for their methods, as yet, I have a quite other feeling. The interest of this problem alone is sufficient to justify the craving of the pure mathematician for powerful methods and unexceptionable rigor.

NON-EUCLIDIAN GEOMETRY

But I turn to another matter. It is an old view, I suppose, that geometry deals with facts about which there can be no two opinions. You are familiar with the axiom that, given a straight line and a point, one and only one straight line can be drawn through the point parallel to the given straight line. According to the old view the natural man would say that this is either true or false. And, indeed, many and long were the attempts made to justify it. At length there came a step which to many probably will still seem unintelligible. A system of geometry was built up in which it is assumed that, given a straight line and a point, an infinite number of straight lines can be drawn through the point, in the plane of the given line, *no one* of which meets the given line. Can there then, one asks at first, be two systems of geometry, both of which are true, though they differ in such an important particular? Almost as soon believe that there can be two systems of laws of nature, essentially differing in character, both reducing the phenomena we observe to order and system—a monstrous heresy, of course! I will only say that, after a century of discussion we are quite sure that many systems of geometry are possible, and true; though not all may be expedient. And if you reply that a geometry is useful for life only in proportion as it fits the properties of concrete things, I will answer, first, are the heavens not then concrete? And have we as yet any geometry that enables us to form a consistent logical idea of furthermost space? And, second, that the justification of such speculations is the interest they evoke, and that the investigations already undertaken have yielded results of the most surprising interest.

THE THEORY OF GROUPS

To-day we characterize a geometry by the help of another general notion, also, for the most part, elaborated in the last hundred years, by means of its group. A group is a set of operations which is closed, in the same sense that the performance of any two of these operations in succession is equivalent to another operation of the set, just as the result of two successive movements of a rigid body can be achieved by a single movement. One of the earliest conscious applications of the notion was in the problem of solving algebraic equations by means of equations of lower order. An equation of the fourth order can be solved by means of a cubic equation, because there exists a rational function of the four roots which takes only three values when the roots are exchanged in all possible ways. Following out this suggestion for an equation of any order, we are led to consider, taking any particular rational function of its roots, what is the group of interchanges among them which leaves this function unaltered in value. This group characterizes the function, all other rational functions unaltered by the same group of interchanges being expressible rationally in terms of this function. On these lines a complete theory of equations which are soluble algebraically can be given. Any

one who wishes to form some idea of the richness of the landscape offered by pure mathematics might do worse than make himself acquainted with this comparatively small district of it. But the theory of groups has other applications. It may be interesting to refer to the circumstance that the group of interchanges among four quantities which leave unaltered the product of their six differences is exactly similar to the group of rotations of a regular tetrahedron whose center is fixed, when its corners are interchanged among themselves. Then I mention the historical fact that the problem of ascertaining when that well-known linear differential equation called the hypergeometric equation has all its solutions expressible in finite terms as algebraic functions, was first solved in connection with a group of similar kind. For any linear differential equation it is of primary importance to consider the group of interchanges of its solutions when the independent variable, starting from an arbitrary point, makes all possible excursions, returning to its initial value. And it is in connection with this consideration that one justification arises for the view that the equation can be solved by expressing both the independent and dependent variables as single-valued functions of another variable. There is, however, a theory of groups different from those so far referred to, in which the variables can change continuously; this alone is most extensive, as may be judged from one of its lesser applications, the familiar theory of the invariants of quantics. Moreover, perhaps the most masterly of the analytical discussions of the theory of geometry has been carried through as a particular application of the theory of such groups.

THE THEORY OF ALGEBRAIC FUNCTIONS

If the theory of groups illustrates how a unifying plan works in mathematics beneath bewildering detail, the next matter I refer to well shows what a wealth, what a grandeur, of thought may spring from what seem slight beginnings. Our ordinary integral calculus is well-nigh powerless when the result of integration is not expressible by algebraic or logarithmic functions. The attempt to extend the possibilities of integration to the case when the function to be integrated involves the square root of a polynomial of the fourth order, led first, after many efforts, among which Legendre's devotion of forty years was part, to the theory of doubly-periodic functions. To-day this is much simpler than ordinary trigonometry, and, even apart from its applications, it is quite incredible that it should ever again pass from being among the treasures of civilized man. Then, at first in uncouth form, but now clothed with delicate beauty, came the theory of general algebraical integrals, of which the influence is spread far and wide; and with it all that is systematic in the theory of plane curves, and all that is associated with the conception of a Riemann surface. After this came the theory of multiply-periodic functions of any number of variables, which, though still very far indeed from being complete, has, I have always felt, a majesty of conception which is unique. Quite recently the ideas evolved in the previous history have prompted a vast general theory of the classification of algebraical surfaces according to their essential properties, which is opening endless new vistas of thought.

THEORY OF FUNCTIONS OF COMPLEX VARIABLES: DIFFERENTIAL EQUATIONS

But the theory has also been prolific in general principles for functions of complex variables. Of greater theories, the problem of automorphic functions alone is a

vast continent still largely undeveloped, and there is the incidental problem of the possibilities of geometry of position in any number of dimensions, so important in so many ways. But, in fact, a large proportion of the more familiar general principles, taught to-day as theory of functions, have been elaborated under the stimulus of the foregoing theory. Besides this, however, all that precision of logical statement of which I spoke at the beginning is of paramount necessity here. What exactly is meant by a curve of integration, what character can the limiting points of a region of existence of a function possess, how even best to define a function of a complex variable, these are but some obvious cases of difficulties which are very real and pressing to-day. And then there are the problems of the theory of differential equations. About these I am at a loss what to say. We give a name to the subject, as if it were one subject, and I deal with it in the fewest words. But our whole physical outlook is based on the belief that the problems of nature are expressible by differential equations; and our knowledge of even the possibilities of the solutions of differential equations consists largely, save for some special types, of that kind of ignorance which, in the nature of the case, can form no idea of its own extent. There are subjects whose whole content is an excuse for a desired solution of a differential equation; there are infinitely laborious methods of arithmetical computation held in high repute of which the same must be said. And yet I stand here to-day to plead with you for tolerance of those who feel that the prosecution of the theoretic studies, which alone can alter this, is a justifiable aim in life! Our hope and belief is that over this vast domain of differential equations the theory of functions shall one day rule, as already it largely does, for example, over linear differential equations.

THEORY OF NUMBERS

In concluding this table of contents, I would also refer, with becoming brevity, to the modern developments of theory of numbers. Wonderful is the fascination and the difficulty of these familiar objects of thought—ordinary numbers. We know how the great Gauss, whose lynx eye was laboriously turned upon all the physical science of his time, has left it on record that in order to settle the law of a plus or minus sign in one of the formulæ of his theory of numbers he took up the pen every week for four years. In these islands perhaps our imperial necessities forbid the hope of much development of such a theoretical subject. But in the land of Kummer and Gauss and Dirichlet the subject to-day claims the allegiance of many eager minds. And we can reflect that one of the latest triumphs has been with a problem known by the name of our English senior wrangler, Waring—the problem of the representation of a number by sums of powers.

Ladies and gentlemen, I have touched only a few of the matters with which pure mathematics is concerned. Each of those I have named is large enough for one man's thought; but they are interwoven and interlaced in indissoluble fashion and form one mighty whole, so that to be ignorant of one is to be weaker in all. I am not concerned to depreciate other pursuits, which seem at first sight more practical; I wish only, indeed, as we all do, it were possible for one man to cover the whole field of scientific research; and I vigorously resent the suggestion that those who follow these studies are less careful than others of the urgent needs of our national life. But

pure mathematics is not the rival, even less is it the handmaid, of other branches of science. Properly pursued, it is the essence and soul of them all. It is not for them; they are for it; and its results are for all time. No man who has felt its fascination can be content to be ignorant of any manifestation of regularity and law, or can fail to be stirred by all the need of adjustment of our actual world.

And if life is short, if the greatest magician, joining with the practical man, reminds us that, like this vision,

> The cloud-capp'd towers, the gorgeous palaces,
> The solemn temples, the great globe itself,
> Yea, all which it inherit, shall dissolve
> And . . . leave not a rack behind,

we must still believe that it is best for us to try to reach the brightest light. And all here must believe it; for else—no fact is more firmly established—we shall not study science to any purpose.

But that is not all I want to say, or at least to indicate. I have dealt so far only with proximate motives; to me it seems demonstrable that a physical science that is conscientious requires the cultivation of pure mathematics; and the most mundane of reasons seem to me to prompt the recognition of the esthetic outlook as a practical necessity, not merely a luxury, in a successful society. Nor do I want to take a transcendental ground. Every schoolboy, I suppose, knows the story of the child born so small, if I remember aright, that he could be put into a quart pot, in a farmhouse on the borders of Lincolnshire—it was the merest everyday chance. By the most incalculable of luck his brain-stuff was so arranged, his parts so proportionately tempered, that he became Newton, and taught us the laws of the planets. It was the blindest concurrence of physical circumstances; and so is all our life. Matter in certain relations to itself, working by laws we can examine in the chemical laboratory, produces all these effects, produces even that state of brain which accompanies the desire to speak of the wonder of it all. And the same laws will inevitably hurl all into confusion and darkness again; and where will all our joys and fears, and all our scientific satisfaction, be then?

As students of science, we have no right to shrink from this point of view; we are pledged to set aside prepossession and dogma, and examine what seems possible, wherever it may lead. Even life itself may be mechanical, even the greatest of all things, even personality, may some day be resoluble into the properties of dead matter, whatever that is. We can all see that its coherence rises and falls with illness and health, with age and physical conditions. Nor, as it seems to me, can anything but confusion of thought arise from attempts to people our material world with those who have ceased to be material.

An argument could perhaps be based on the divergence, as the mathematician would say, of our comprehension of the properties of matter. For though we seem able to summarize our past experiences with ever-increasing approximation by means of fixed laws, our consciousness of ignorance of the future is only increased thereby. Do we feel more, or *less,* competent to grasp the future possibilities of things, when we can send a wireless message 4,000 miles, from Hanover to New Jersey?

Our life is begirt with wonder, and with terror. Reduce it by all means to ruthless mechanism, *if you can;* it will be a great achievement. But it can make no sort of difference to the fact that the things for which we live are spiritual. The rose is no less sweet because its sweetness is conditioned by the food we supply to its roots.

It is an obvious fact, and I ought to apologize for remarking it, were it not that so much of our popular science is understood by the haste to imply an opposite conclusion. If a chemical analysis of the constituents of sea-water could take away from the glory of a mighty wave breaking in the sunlight, it would still be true that it was the mind of the chemist which delighted in finding the analysis. Whatever be its history, whatever its physical correlations, it is an undeniable fact that the mind of man has been evolved; I believe that is the scientific word. You may speak of a continuous upholding of our material framework from without; you may ascribe fixed qualities to something you call matter; or you may refuse to be drawn into any statement. But anyway, the fact remains that the precious things of life are those we call the treasures of the mind. Dogmas and philosophies, it would seem, rise and fall. But gradually accumulating throughout the ages, from the earliest dawn of history, there is a body of doctrine, a reasoned insight into the relations of exact ideas, painfully won and often tested. And this remains the main heritage of man; his little beacon of light amidst the solitudes and darknesses of infinite space; or, if you prefer, like the shout of children at play together in the cultivated valleys, which continues from generation to generation.

Yes, and continues for ever! A universe which has the potentiality of becoming thus conscious of itself is not without something of which that which we call memory is but an image. Somewhere, somehow, in ways we dream not of, when you and I have merged again into the illimitable whole, when all that is material has ceased, the faculty in which we now have some share shall surely endure; the conceptions we now dimly struggle to grasp, the joy we have in the effort, these are but part of a greater whole. Some may fear, and some may hope, that they and theirs shall not endure for ever. But he must have studied nature in vain who does not see that our spiritual activities are inherent in the mighty process of which we are part; who can doubt of their persistence.

And, on the intellectual side, of all that is best ascertained, and surest, and most definite, of these; of all that is oldest and most universal; of all that is most fundamental and far-reaching, of these activities, pure mathematics is the symbol and the sum.

H. F. BAKER

WORK GOING ON AT KILAUEA VOLCANO

FOR the past three months the "fires of Pelé" have been comparatively low, the conditions in the active pit of Halemaumau being that of unusual quiescence. The level of the bottom has also remained lower than at any other time since last fall, when it had a depth of about 700 feet. The last plane-table measurement obtained gave a depth of 550 feet and the subsequent change has been small.

Since its late maximum height of about 350 feet below the rim, in January of this year, the liquid lava lake has in its general movement been dropping, though a rise in June and July, 1912, presented an activity greater than any other recorded during the past thirty years. There was a molten lake 650 feet long by 450 feet wide. As many as six hundred fountains of liquid magma played simultaneously and threw the molten spray to heights of twenty to thirty feet, accompanied by a sound like the roar of heavy ocean surf. In December, after intermittent declines, the level of the lake again rose, though to a lesser altitude and accompanied with decreased activity; this condition continuing until the middle of February of this year. Since then the resultant effects have been a lowering of the surface of the lava column until, on May 1, it disappeared from all view either by day or through incandescence at night.